

# Greedy Job Selection

Wednesday, 8 September 2021 2:03 PM

Problem: Given matroid  $M = (S, \mathbb{I})$  with wt.  $w(s_i)$  on each element  $s_i \in S$ , find an independent set of max. wt. All wts. are nonnegative.

Algo: Sort elts. so that  $w(s_1) \geq w(s_2) \geq \dots \geq w(s_n)$

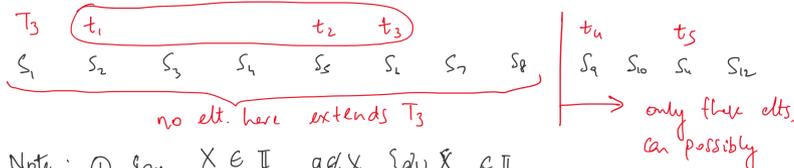
$T \leftarrow \emptyset$

for  $i = 1 \dots n$

if  $T \cup \{s_i\} \in \mathbb{I}$ ,  $T \leftarrow T \cup \{s_i\}$

Claim:  $T$  obtained at the end of the algo is an IS of max. wt.

Proof: let  $T = \{t_1, t_2, t_3, \dots, t_k\}$  in decreasing order of wt.



Note: ① say  $X \in \mathbb{I}$ ,  $a \notin X$ ,  $\{a\} \cup X \in \mathbb{I}$ .

Then for any  $Y \subseteq X$ ,  $\{a\} \cup Y \in \mathbb{I}$

② no elt. before  $t_i$  extends  $T_{i-1}$  (in example above  $i=4$ )

Let  $T_i = \{t_1, t_2, t_3, \dots, t_i\}$ , i.e.,  $T_i$  is the first  $i$  elts of  $T$ .

Actually will show that for all  $i \leq k$ ,  $T_i$  is the highest wt. IS of size  $i$ .

Base case: clearly true for  $i=0$

Suppose  $T_{i-1}$  is the max wt. IS of size  $i-1$

For a contradiction, assume  $w(\hat{T}_i) > w(T_i)$

By exchange property,  $\exists a \in \hat{T}_i \setminus T_{i-1}$  s.t.  $T_{i-1} \cup \{a\} \in \mathbb{I}$ .

But then,  $a$  must lie after  $t_i$ , or  $w(a) \leq w(t_i)$  by Note 2

also,  $w(\hat{T}_i \setminus \{a\}) \leq w(T_{i-1})$  by induction hypothesis

Thus  $w(\hat{T}_i) \leq w(T_{i-1} \cup t_i)$

$= w(T_i)$   $\blacksquare$

Do yourself: Now suppose some elements have negative wt.

How do you modify algorithm?

OR: if you want max wt. IS of size  $k$ , in presence of -ve wts.?

OR: if you want min wt. IS?

## Problem: Job Selection

Given:  $n$  jobs  $J$

deadlines  $d_1, d_2, d_3, \dots, d_n$

penalties  $w_1, w_2, w_3, \dots, w_n$

Each job takes 1 unit of time to finish processing.

We have a single machine

Defn: Given  $S \subseteq J$ , a schedule is an ordering

of the jobs in  $S$

Given  $S \subseteq J$  a feasible schedule is an ordering

where each job completes <sup>on or</sup> before its deadline

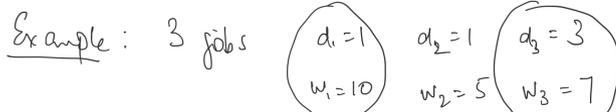
Given  $S \subseteq J$ ,  $S$  is a feasible set if  $S$

has a feasible schedule

Problem: Find a feasible set  $S \subseteq J$  that

$$\text{minimizes } \sum_{j \notin S} w_j = \sum_{j \in J} w_j - \sum_{j \in S} w_j$$

maximize this



Let  $D := \max d_j$

For any  $t \leq D$ , given  $S \subseteq J$ ,  $N_t(S) = \#$  of jobs in  $S$  that

have deadlines  $\leq t$

$$= \{j \in S : d_j \leq t\}$$

Lemma: Given  $S \subseteq J$ ; the following statements are

equivalent:

①  $S$  has a feasible schedule

② For any  $t \leq D$ ,  $N_t(S) \leq t$

③ The schedule that orders jobs in  $S$  by increasing deadlines

is feasible

Proof: ③  $\Rightarrow$  ① : trivial

①  $\Rightarrow$  ② : by contradiction

②  $\Rightarrow$  ③ : easy. (do yourself)

Lemma: Let  $\mathbb{I} = \{S \subseteq J : S \text{ is feasible}\}$ . Then

$M = (J, \mathbb{I})$  is a matroid.

Proof: ① Downward-closed is trivial, if  $S \subseteq J$  is feasible

then so is any subset

② Let  $S, T \subseteq J$  s.t. both are feasible &  $|S| > |T|$ .

We assume wlog that no two jobs have same deadline

deadline

(eg. we could index the jobs & use the index as a tiebreaker).

Let  $\hat{j}$  be the job in  $S$  w/ largest deadline that is not in  $T$ .

i.e.,  $\hat{j} = \arg \max_{j \in S \setminus T} d_j$

Then we claim that  $T \cup \{\hat{j}\} \in \mathbb{I}$ , to complete the proof.

Claim:  $T \cup \{\hat{j}\} \in \mathbb{I}$

Proof: Let  $T^< = \{j \in T : d_j < d_{\hat{j}}\}$ ,  $T^> = \{j \in T : d_j > d_{\hat{j}}\}$

&  $S^<$ ,  $S^>$  defined similarly.

By definition,  $T^> = S^>$ .

Since  $|S| > |T|$ ,  $|S^<| \geq |T^<|$ .

(the weak inequality is since  $\hat{j} \in S \setminus T$ )

If  $T \cup \{\hat{j}\}$  is not feasible,  $\exists t^*$  s.t.  $N_{t^*}(T \cup \{\hat{j}\}) > t^*$ .

Since  $T$  is feasible,  $t^* \geq d_{\hat{j}}$ .

But then  $N_{t^*}(T \cup \{\hat{j}\}) = |\{j \in T \cup \{\hat{j}\} : d_j \leq t^*\}|$

$$= |T^<| + 1 + |\{j \in T \cup \{\hat{j}\} : d_j < d_{\hat{j}} \leq t^*\}|$$

$$\leq |S^<| + 1 + |\{j \in S : d_j < d_{\hat{j}} \leq t^*\}|$$

$$= N_{t^*}(S)$$

Since  $S$  is feasible,  $N_{t^*}(S) \leq t^*$ , giving a contradiction  $\blacksquare$

(Diff proof. as pointed out in class:

consider the laminar matroid where  $S \in \mathbb{I}$  iff  $N_t(S) \leq t$ .

clearly,  $S \in \mathbb{I}$  iff  $S$  is feasible),