Greedy Job Selection

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Problem: Given matroid $M = (S, \mathcal{I})$ with wt. $w(s_i)$ on each element $s_i \in S$, find an independent set of max. wt. All wts. are nonnegative.

Algo: Sort elts. so that $w(s_1) \geq w(s_2) \geq \dots \geq w(s_n)$

$T \leftarrow \phi$

for $i = 1 \dots n$

if $\boxed{T \cup \{s_i\} \in \mathcal{I}}$, $T \leftarrow T \cup \{s_i\}$

Claim: T obtained at the end of the algo is an IS of max. wt.

Proof: Let $T = \{t_1, t_2, t_3, \dots, t_k\}$ in decreasing order of wt.

T_3 : t_1 t_2 t_3 | t_4 t_5

s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 | s_9 s_{10} s_{11} s_{12}

$\underbrace{s_1 \dots s_8}_{\text{no elt. here extends } T_3}$ → only these elts. can possibly extend T_3

Note: ① Say $X \in \mathcal{I}$, $a \notin X$, $\{a\} \cup X \in \mathcal{I}$. Then for any $Y \subseteq X$, $\{a\} \cup Y \in \mathcal{I}$

② no elt. before t_i extends T_{i-1} (in example above $i = 4$)

Let $T_i = \{t_1, t_2, t_3, \dots, t_i\}$, i.e., T_i is the first i elts of T.

Actually will show that for all $i \leq k$, T_i is the highest wt. IS of size i.

Base case: clearly true for $i = 0$

Suppose T_{i-1} is the max wt. IS of size $i-1$

For a contradiction, assume $w(\hat{T}_i) > w(T_i)$

By exchange property, $\exists\, a \in \hat{T}_i \setminus T_{i-1}$ s.t. $T_{i-1} \cup \{a\} \in \mathcal{I}$.

But then, a must lie after t_i, or $w(a) \leq w(t_i)$ by Note 2

also, $w(\hat{T}_i \setminus a) \leq w(T_{i-1})$ by Induction hypothesis

Thus $w(\hat{T}_i) \leq \underline{w(T_{i-1} \cup t_i)}$

$= w(T_i)$ ∎

Do yourself: Now suppose some elements have negative wt. How do you modify algorithm?

OR: if you want max wt. IS of size k, in presence of $-$ve wts.?

OR: if you want min wt. IS?

Problem: Job Selection

Given: n jobs J

deadlines d_1 d_2 d_3 $\dots$ d_n

penalties w_1 w_2 w_3 $\dots$ w_n

Each job takes 1 unit of time to finish processing.

We have a single machine

Defn: Given $S \subseteq J$, a <u>schedule</u> is an ordering of the jobs in S

Given $S \subseteq J$ a <u>feasible schedule</u> is an ordering where each job completes on or before its deadline

Given $S \subseteq J$, S is a <u>feasible set</u> if S has a feasible schedule

Problem: Find a feasible set $S \subseteq J$ that minimizes $\sum_{j \notin S} w_j = \sum_{j \in J} w_j - \underbrace{\sum_{j \in S} w_j}_{\text{maximize this}}$

Example: 3 jobs

$d_1 = 1$	$d_2 = 1$	$d_3 = 3$
$w_1 = 10$	$w_2 = 5$	$w_3 = 7$

Let $D := \max_j d_j$

For any $t \leq D$, given $S \subseteq J$, $N_t(S)$ = # of jobs in S that have deadlines $\leq t$

$= \{j \in S : d_j \leq t\}$

Lemma: Given $S \subseteq J$; the following statements are equivalent:

(i) S has a feasible schedule

(ii) For any $t \leq D$, $N_t(S) \leq t$

(iii) The schedule that orders jobs in S by increasing deadlines is feasible

Proof: (iii) $\Rightarrow$ (i) : trivial

(i) $\Rightarrow$ (ii) : by contradiction

(ii) $\Rightarrow$ (iii) : easy. (do yourself)

Lemma: Let $\mathcal{I} = \{S \subseteq J : S \text{ is feasible}\}$. Then $M = (J, \mathcal{I})$ is a matroid.

Proof: (I) Downward-closed is trivial, if $S \subseteq J$ is feasible then so is any subset

(II) Let $S, T \subseteq J$ s.t. both are feasible & $|S| > |T|$.

We assume wlog that no two jobs have same deadline

(eg. we could index the jobs & use the index as a tiebreaker).

Let $\hat{j}$ be the job in S w/ largest deadline that is not in T.

i.e., $\hat{j} = \arg\max_{j \in S \setminus T} d_j$

Then we claim that $T \cup \{\hat{j}\} \in \mathcal{I}$, to complete the proof.

Claim: $T \cup \{\hat{j}\} \in \mathcal{I}$

Proof: Let $T^{<} = \{j \in T : d_j < d_{\hat{j}}\}$, $T^{>} = \{j \in T : d_j > d_{\hat{j}}\}$

& $S^{<}$, $S^{>}$ defined similarly.

By definition, $T^{>} = S^{>}$.

Since $|S| > |T|$, $|S^{<}| \geq |T^{<}|$.

(the weak inequality is since $\hat{j} \in S \setminus T$)

If $T \cup \{\hat{j}\}$ is not feasible, $\exists\, t^*$ s.t. $N_{t^*}(T \cup \hat{j}) > t^*$.

Since T is feasible, $t^* \geq d_{\hat{j}}$.

But then $N_{t^*}(T \cup \hat{j}) = |\{j \in T \cup \hat{j} : d_j \leq t^*\}|$

$= |T^{<}| + 1 + |\{j \in T \cup \hat{j} : d_{\hat{j}} < d_j \leq t^*\}|$

$\leq |S^{<}| + 1 + |\{j \in S : d_{\hat{j}} < d_j \leq t^*\}|$

$= N_{t^*}(S)$

Since S is feasible, $N_{t^*}(S) \leq t^*$, giving a contradiction ∎

(Diff proof. as pointed out in class:

Consider the laminar matroid where $S \in \mathcal{I}$ iff $N_t(S) \leq t$.

clearly, $S \in \mathcal{I}$ iff S is feasible).